JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **144**, 109–127 (1989)

On the Nonlinear Stability of the Rotating Bénard Problem via the Lyapunov Direct Method

G. MULONE

Dipartimento di Matematica,
Viale A. Doria 6, 95125 Catania, Italy

AND

S. RIONERO

Dipartimento di Matematica e Applicazioni,
Via Mezzocannone 8, 80134 Naples, Italy

Submitted by Cathleen S. Morawetz

Received August 13, 1987

In this paper we use the Lyapunov direct method to study the nonlinear conditional stability of the Bénard problem with rotation. In particular, for Prandtl numbers greater than or equal to one, and for Taylor numbers less than or equal to $80\pi^4$, we prove the coincidence between the linear and nonlinear critical stability parameters. We also give some values of the attracting radius for the conditionally stable disturbances of the basic motion. © 1989 Academic Press, Inc.

1. INTRODUCTION

This paper is dedicated to the study of the nonlinear stability of the rotating Bénard problem [1–14]. As is well known [3], the most important phenomenon in the rotating Bénard problem is the inhibiting effect of rotation on the onset of convection. This effect has been proved by Chandrasekhar [3] in the framework of linear stability. Successively it has been experimentally verified by Rossby [11]. In the framework of nonlinear stability, we recall the results of Veronis [14] for two-dimensional disturbances, and—recently—the results of Galdi and Straughan [5], which, although close to those for linear stability (in particular range of the Taylor number), are generally below them.

In the present paper, using the Lyapunov direct method [5, 15–19] and following the guideline proposed in [17–18], we are able to prove the coincidence between the conditions of linear and nonlinear stability in a

0022-247X/89 $3.00

409/144/1-8

suitable interval of the Taylor number (under a restriction on the initial data) and obtain, in any case, results in agreement (in a restricted sense, see [5, p. 277]) with the experiments of [11] (which were done for rigid boundary conditions).

The plan of the paper is as follows: In Section 2 the basic equations are presented. In Section 3 we choose the Lyapunov function V. In Sections 4 and 5 we give a nonlinear stability theorem. In Section 6 we solve a variational problem for the nonlinear critical stability parameter (Rayleigh number). Then for Prandtl numbers greater than 1 we prove, in Section 7 the coincidence between the linear and nonlinear (conditional) critical stability parameters for Taylor numbers less or equal to $80\pi^4$. In Section 8 we investigate under which conditions on the initial data the perturbations are attracted to the basic motion. The paper ends with some final comments (Section 9) and an Appendix.

2. Equation of Perturbation

Let us consider an infinite horizontal layer of a homogeneous fluid under the action of a vertical gravity field $\mathbf{g} = -g\mathbf{k}$ in which an adverse temperature gradient $\beta > 0$ is maintained. Moreover, let us assume that the fluid is rotating about the vertical axis with a constant angular velocity Ω and let $0xyz = (0, \mathbf{i}, \mathbf{j}, \mathbf{k})$ be the frame of reference rotating about z with the same angular velocity. Here we study the stability of a nonconvecting stationary solution $m_0 = (\hat{\mathbf{v}} = 0,\ \hat{T} = -\beta z + \hat{T}_0,\ \hat{p})$. The fluid is confined between the planes $z = 0$ and $z = d > 0$ with assigned temperatures $\hat{T}(z=0) = \hat{T}_0$, $\hat{T}(z=d) = -\beta d + \hat{T}_0$.

The Boussinesq equations for a perturbation $(\mathbf{u}, \vartheta, p)$ to m_0 are given in [3, Section 25, p. 87]. Here, we use the nondimensional form given in [5],

$$\begin{aligned} &\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + R\vartheta\mathbf{k} + \Delta\mathbf{u} + T\mathbf{u}\times\mathbf{k} \\ &P_r\vartheta_t + P_r\mathbf{u}\cdot\nabla\vartheta = Rw + \Delta\vartheta \\ &\nabla\cdot\mathbf{u} = 0, \end{aligned} \tag{2.1}$$

where $\mathbf{u} = (u, v, w)$, ϑ, p represent the (non-dimensional) perturbation velocity, temperature, and pressure field (incorporating centrifugal forces), and P_r, R^2, T^2 are the Prandtl, Rayleigh, and Taylor numbers. It is understood that the spatial region for (2.1) is now $\mathbb{R}^2 \times [0, 1] = \hat{\Omega}$.

With the system (2.1) we adopt the initial and boundary conditions

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \qquad \vartheta(\mathbf{x}, 0) = \vartheta_0(\mathbf{x}), \qquad \mathbf{x} \in \hat{\Omega} \tag{2.2}$$

$$w(\mathbf{x}, t) = \vartheta(\mathbf{x}, t) = 0, \qquad u_z(\mathbf{x}, t) = v_z(x, t) = 0, \qquad (\mathbf{x}, t) \in \partial\hat{\Omega} \times [0, \infty) \tag{2.3}$$

where $\mathbf{x}=(x, y, z)$, $f_z=\partial f/\partial z$, and $\vartheta_0(\mathbf{x})$, $\mathbf{u}_0(\mathbf{x})$ are assigned fields with $\nabla \cdot \mathbf{u}_0(\mathbf{x})=0$. The conditions (2.3) are the typical ones of the stress-free boundary case [3, pp. 21–22].

Here, we assume that the perturbation fields are periodic functions of x and y of periods $2\pi/a_x$, $2\pi/a_y$, respectively ($a_x>0$, $a_y>0$), and we denote by Ω_1 the periodicity cell

$$\Omega_1=\left[0, \frac{2\pi}{a_x}\right]\times\left[0, \frac{2\pi}{a_y}\right]\times[0, 1]$$

and by $a=(a_x^2+a_y^2)^{1/2}$ the wave number. Moreover, taking into account that the stability of the solution $m_0=(\mathbf{u}=0, \vartheta=0)$ makes sense only in a class of solutions of (2.1), (2.3) in which m_0 is unique, we exclude any other rigid solution requiring the "average velocity condition" (see [7])

$$\int_{\Omega_1} u\, d\Omega_1=\int_{\Omega_1} v\, d\Omega_1=0. \tag{2.4}$$

3. Essential Variables and Choice of Lyapunov Function

Let us recall that the variables used in the linear stability or rotating Bénard problem (RBP) are the vertical components of velocity and vorticity and the temperature,

$$w, \zeta, \vartheta, \tag{3.1}$$

where

$$\zeta=\mathbf{k}\cdot\nabla\times\mathbf{u} \tag{3.2}$$

(see [3], pp. 88–89). Following [17–18], we call the variables (3.1) the *essential variables* of the RBP. As "balance" of causes which may promote or inhibit the instability, we choose the function

$$f=\zeta-c\vartheta_z, \qquad (c>0), \tag{3.3}$$

and we use the Lyapunov function

$$V(t)=V_0(t)+bV_1(t), \qquad (b>0), \tag{3.4}$$

where

$$\begin{aligned} V_0(t)&=\tfrac{1}{2}[\|\nabla w\|^2+a_1\|\vartheta_z\|^2+a_2\|f\|^2]\\ V_1(t)&=\tfrac{1}{2}[\|\nabla\mathbf{u}\|^2+P_r\|\nabla\vartheta\|^2] \end{aligned} \tag{3.5}$$

with

$$\|\cdot\| = L_2(\Omega_1)\text{-norm}.$$

In (3.4) and (3.5) the positive constants b, a_1, a_2, c are the Lyapunov parameters and they will be chosen later.

In order to evaluate the time derivative $\dot{V}$ along the solutions to the perturbation equations, we need the evolution equations governing the behaviour of ζ, Δw, ϑ_z, f, and consequently further boundary conditions.

Let us apply the operator $\nabla\times$ to the equation $(2.1)_1$. The result is

$$(\nabla\times\mathbf{u})_t + \nabla\times(\mathbf{u}\cdot\nabla\mathbf{u}) = R\nabla\times(\vartheta\mathbf{k}) + \Delta(\nabla\times\mathbf{u}) + T\nabla\times(\mathbf{u}\times\mathbf{k}). \qquad (3.6)$$

Taking the $\nabla\times$ of the last equation once again, we have

$$\begin{aligned}&[\nabla\times(\nabla\times\mathbf{u})]_t + \nabla\times\nabla\times(\mathbf{u}\cdot\nabla\mathbf{u})\\ &\quad = R\nabla\times(\nabla\times\vartheta\mathbf{k}) + \Delta(\nabla\times\nabla\times\mathbf{u}) + T\nabla\times\nabla\times(\mathbf{u}\times\mathbf{k}).\end{aligned} \qquad (3.7)$$

From (3.6) and (3.7) the evolution equations of ζ and w follow (see [3, p. 88]):

$$\begin{aligned}\zeta_t &= Tw_z + \Delta\zeta - \nabla\times(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{k}\\ \Delta w_t &= R\Delta_1\vartheta + \Delta\Delta w - T\zeta_z + \nabla\times[\nabla\times(\mathbf{u}\cdot\nabla\mathbf{u})]\cdot\mathbf{k}.\end{aligned} \qquad (3.8)$$

From $(2.1)_2$ and $(3.8)_1$ the evolution equations for ϑ_z and f follow:

$$\begin{aligned}\vartheta_{zt} &= \frac{R}{P_r}w_z + \frac{1}{P_r}\Delta\vartheta_z - \mathbf{u}_z\cdot\nabla\vartheta - \mathbf{u}\cdot\nabla\vartheta_z\\ f_t &= \left(T - \frac{cR}{P_r}\right)w_z + \Delta f + c\left(1 - \frac{1}{P_r}\right)\Delta\vartheta_z\\ &\quad - [\nabla\times(\mathbf{u}\cdot\nabla\mathbf{u})]\cdot\mathbf{k} + c\mathbf{u}_z\cdot\nabla\vartheta + c\mathbf{u}\cdot\nabla\vartheta_z.\end{aligned} \qquad (3.9)$$

The further boundary conditions that we need are

$$\zeta_z = w_{zz} = \Delta w = \vartheta_{zz} = f_z = 0 \qquad \text{on } z=0, \quad z=1, \qquad (3.10)$$

and are obtained as follows. The conditions $(3.10)_1$ and $(3.10)_2$ are contained in [3, p. 22]. On the other hand $(2.3)_2$ implies

$$\vartheta_t = \vartheta_x = \vartheta_y = \vartheta_{xx} = \vartheta_{yy} = 0 \qquad \text{on} \quad z=0, \quad z=1. \qquad (3.11)$$

(2.1), together with $(2.3)_1$ and (3.11), gives

$$\vartheta_{zz} = 0 \qquad \text{on} \quad z=0, z=1. \qquad (3.12)$$

Moreover, $(2.3)_1$ implies

$$\Delta_1 w = 0 \qquad \text{on} \quad z=0, \quad z=1. \tag{3.13}$$

Then (3.13) with $(3.10)_2$ gives $(3.10)_3$. Finally $(3.10)_5$ follows from $(3.10)_1$, $(3.10)_4$, and (3.3).

Remark 1. From the mathematical point of view the study of the stability through the function (3.4)–(3.5) is equivalent to the study of the stability of m_0 in the norm of the Sobolev space $W_2^1(\Omega_1)$. In fact, taking into account the Poincaré inequalities

$$\begin{aligned} &\pi^2 \|\mathbf{u}\|^2 \leqslant \|\nabla \mathbf{u}\|^2 \\ &\pi^2 \|\vartheta\|^2 \leqslant \|\nabla \vartheta\|^2, \end{aligned} \tag{3.14}$$

we have

$$\begin{aligned} &\frac{b}{2} \frac{1}{\max(1, 1/P_r)} \{ \|\mathbf{u}\|^2 + \|\vartheta\|^2 + \|\nabla \mathbf{u}\|^2 + \|\nabla \vartheta\|^2 \} \leqslant V \\ &\quad \leqslant \max(1, 1/2 + 4a_2 + b/2, a_1 + 2a_2 c^2 + bP_r/2) \\ &\quad \times \{ \|\mathbf{u}\|^2 + \|\vartheta\|^2 + \|\nabla \mathbf{u}\|^2 + \|\nabla \vartheta\|^2 \}. \end{aligned} \tag{3.15}$$

As is well known, the choice of the W_2^1-norm has been useful in problems of fluid stability [19–23].

4. A Priori Estimate of $\dot{V}$

In order to study nonlinear stability by the Lyapunov direct method we must calculate the time derivative of the Lyapunov function V along the solution to perturbation equations. Multiplying scalarly $(3.8)_2$, $(3.9)_1$, $(3.9)_2$, $(2.1)_1$, and $(2.1)_2$ by $-w$, $a_1 \vartheta_z$, $a_2 f$, $-\Delta \mathbf{u}$, and $-\Delta \vartheta$, respectively, integrating over Ω_1, we obtain

$$\dot{V} = I_0 - D_0 + N_0 + bI_1 - bD_1 + bN_1 + B, \tag{4.1}$$

with

$$\begin{aligned} I_0 = &-R(\Delta_1 \vartheta, w) + \left[a_2 \left(T - \frac{cR}{P_r} \right) - T \right] (\zeta, w_z) \\ &+ \frac{a_1 R}{P_r} (w_z, \vartheta_z) - a_2 c \left(T - \frac{cR}{P_r} \right) (w_z, \vartheta_z) - a_2 c \left(1 - \frac{1}{P_r} \right) (\nabla \vartheta_z, \nabla f) \end{aligned}$$

$$D_0 = \|\Delta w\|^2 + \frac{a_1}{P_r} \|\nabla \vartheta_z\|^2 + a_2 \|\nabla f\|^2$$

$$\begin{aligned}
N_0 &= -(\nabla \times (\nabla \times (\mathbf{u} \cdot \nabla \mathbf{u})) \cdot \mathbf{k}, w) - a_1(\mathbf{u}_z \cdot \nabla \vartheta, \vartheta_z) - a_2(\nabla \times (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{k}, f) \\
&\quad + a_2 c(\mathbf{u}_z \cdot \nabla \vartheta, f) + a_2 c(\mathbf{u} \cdot \nabla \vartheta_z, f) \\
I_1 &= 2R(\nabla w, \nabla \vartheta) \\
D_1 &= \|\Delta \vartheta\|^2 + \|\Delta \mathbf{u}\|^2 \\
N_1 &= (\mathbf{u} \cdot \nabla \mathbf{u}, \Delta \mathbf{u}) + P_r(\mathbf{u} \cdot \nabla \vartheta, \Delta \vartheta),
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
B = \int_{\partial \Omega_1} & (n \cdot \nabla w_t w - n \cdot \nabla(\Delta w) w + \mathbf{n} \cdot \nabla w \, \Delta w \\
& + T\mathbf{n} \cdot \mathbf{k} \zeta w + \frac{a_1}{P_r} \mathbf{n} \cdot \nabla \vartheta_z \vartheta_z + \mathbf{n} \cdot \mathbf{u} \vartheta_z^2/2 \\
& + a_2 \mathbf{n} \cdot \nabla f f + a_2 c \left(1 - \frac{1}{P_r}\right) \mathbf{n} \cdot \nabla \vartheta_z f + b\mathbf{n} \cdot \nabla \mathbf{u} \cdot \mathbf{u}_t \\
& + b\mathbf{n} \cdot \Delta \mathbf{u} p - bR\mathbf{n} \cdot \nabla w \vartheta + bT[(\mathbf{n} \cdot \nabla \mathbf{u}) v \\
& + (\mathbf{n} \cdot \nabla v) u] + bP_r \mathbf{n} \cdot \nabla \vartheta \vartheta_t + bR\mathbf{n} \cdot \nabla \vartheta w\} \, d\sigma.
\end{aligned} \tag{4.3}$$

$(\cdot, \cdot)$ denotes the scalar product in $L_2(\Omega_1)$ and $\mathbf{n}$ is the outer unit normal vector on $\partial \Omega_1$.

Now we need an a priori estimate of $\dot{V}$ in order to prove a nonlinear stability theorem. For this, the following observations hold true.

(1) Because of periodicity, the boundary conditions (2.3), (3.10)–(3.13), and the fact that $w_t = 0$ on $z = 0$, $z = 1$, it follows that $B = 0$.

(2) Setting

$$M = \max_{\mathscr{H}_1} \frac{I_0}{D_0}, \tag{4.4}$$

where H_1 is the space of admissible functions,

$$\begin{aligned}
\mathscr{H}_1 = \{ & w, \vartheta, \zeta, f \text{ which are periodic in } x \text{ and } y \text{ and} \\
& \text{satisfy } (2.3)_{1-2}, (3.10)\},
\end{aligned} \tag{4.5}$$

we have

$$\dot{V}_0(t) \leqslant (M - 1) D_0 + N_0. \tag{4.6}$$

(3) By use of some Poincaré-type inequalities and imbedding theorems, we have

$$\pi^2 \|\vartheta_z\|^2 \leqslant \pi^2 \|\nabla\vartheta\|^2 \leqslant \|\nabla\vartheta_z\|^2, \qquad \pi^4 \|w\|^2 \leqslant \pi^2 \|\nabla w\|^2 \leqslant \|\Delta w\|^2 \leqslant \|\Delta \mathbf{u}\|^2$$

$$\sup_{\Omega_1} |\vartheta| = C\|\Delta\vartheta\|, \qquad \sup_{\Omega_1} |\mathbf{u}| \leqslant C\|\Delta\mathbf{u}\|, \ \|w_{x_i x_j}\| \leqslant \|\Delta w\|, \tag{4.7}$$

where x_i, x_j $(i, j = 1, 2, 3)$ stand for x, y, z, and C is a computable positive constant depending on Ω_1.

(The proofs of (4.7) are classical. For $(4.7)_{1-2}$ see [24, 5]. $(4.7)_{3\text{-}5}$ are proved in [25, 26]. A value of the constant C is given in [5, (A.15)]).

Therefore from (4.2), (4.7), the Cauchy–Schwarz and the arithmetic–geometric mean inequalities, we obtain the estimate

$$\begin{aligned} bI_1 &= 2bR(\nabla w, \nabla\vartheta) \leqslant bR^2 \|\nabla\vartheta\|^2/\varepsilon + b\varepsilon \|\nabla w\|^2 \\ &\leqslant \frac{bR^2}{\varepsilon\pi^2}\frac{P_r}{a_1} D_0 + \frac{b\varepsilon}{\pi^2} D_1, \qquad (\varepsilon > 0). \end{aligned}$$

Assuming

$$0 < M < 1, \tag{4.8}$$

choosing

$$\varepsilon = \pi^2/2, \qquad b = \frac{a_1\pi^4(1-M)}{4R^2P_r}, \tag{4.9}$$

and defining

$$D_2 = \frac{1-M}{2} D_0 + \frac{b}{2} D_1, \tag{4.10}$$

we have

$$bI_1 \leqslant D_2. \tag{4.11}$$

(4) From (3.4), (3.5), (4.2), (4.7), (4.8), and (4.10) we have

$$N_0 + bN_1 \leqslant AD_2 V^{1/2} \tag{4.12}$$

with

$$\begin{aligned} A = (8/b)^{1/2} C\{1 + (P_r)^{1/2}/2 + [2/(b - bM)]^{1/2} \\ + \left[\frac{a_1 P_r}{b(1-M)}\right]^{1/2} + c\left[\frac{a_2}{b(1-M)}\right]^{1/2} + (2)^{1/2}\left[\frac{a_2}{b(1-M)}\right]^{1/2}\right\}. \end{aligned} \tag{4.13}$$

The proof of (4.12) is achieved as follows (we use the Poincaré inequality and the imbedding (4.7)):

$$\begin{aligned}
bN_1 &= b(\mathbf{u}\cdot\nabla\mathbf{u}, \Delta\mathbf{u}) + bP_r(\mathbf{u}\cdot\nabla\vartheta, \Delta\vartheta)\\
&\leqslant b\sup_{\Omega_1}|\mathbf{u}|\Big(\|\nabla\mathbf{u}\|\,\|\Delta\mathbf{u}\| + P_r\|\nabla\vartheta\|\,\|\Delta\vartheta\|\Big)\\
&\leqslant bC\|\nabla\mathbf{u}\|\,\|\Delta\mathbf{u}\|^2 + P_r\|\nabla\vartheta\|\,\|\Delta\vartheta\|\,\|\Delta\mathbf{u}\|)\\
&\leqslant bC[D_1\|\nabla\mathbf{u}\| + P_rD_1\|\nabla\vartheta\|/2]\\
&\leqslant bC(2)^{1/2}[1+(P_r)^{1/2}/2]\,D_1V_1^{1/2}\\
&\leqslant (8/b)^{1/2}[1+(P_r)^{1/2}/2]\,D_2V^{1/2}.
\end{aligned} \tag{4.14}$$

Moreover we observe that after some integration by parts, because of the boundary condition (2.3), (3.10), and the identity

$$-\nabla\times(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{k} = \nabla\times\mathbf{u}\cdot\nabla w - \mathbf{u}\cdot\nabla\zeta,$$

we have

$$\begin{aligned}
N_0 = &-(\mathbf{u}\cdot\nabla\mathbf{u}, \nabla\times\nabla\times(w\mathbf{k})) - a_1(\mathbf{u}_z\cdot\nabla\vartheta, \vartheta_z)\\
&+ a_2c(\boldsymbol{u}_z\cdot\nabla\vartheta, f) + a_2(\nabla\times\mathbf{u}\cdot\nabla w, f).
\end{aligned}$$

Therefore

$$\begin{aligned}
&-(\mathbf{u}\cdot\nabla\mathbf{u}, \nabla\times\nabla\times(w\mathbf{k}))\\
&\qquad\leqslant (2)^{1/2}C\|\Delta\mathbf{u}\|\,\|\nabla\mathbf{u}\|\,\|\Delta w\| \leqslant 4CD_2V^{1/2}/[b(1-M)^{1/2}],
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
&-a_1(\mathbf{u}_z\cdot\nabla\vartheta, \vartheta_z)\\
&\quad = a_1(\mathbf{u}_z\cdot\nabla\vartheta_z, \vartheta) \leqslant a_1\sup_{\Omega_1}|\vartheta|\,\|\mathbf{u}_z\|\,\|\nabla\vartheta_z\| \leqslant Ca_1\|\Delta\vartheta\|\,\|\nabla\mathbf{u}\|\,\|\nabla\vartheta_z\|\\
&\quad\leqslant \frac{(8)^{1/2}C}{b}\left[\frac{a_1P_r}{1-M}\right]^{1/2} D_2V^{1/2},
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
&a_2c(\mathbf{u}_z\cdot\nabla\vartheta, f)\\
&\quad = -a_2c(\mathbf{u}_z\cdot\nabla f, \vartheta) \leqslant a_2c\sup_{\Omega_1}|\vartheta|\,\|\nabla\mathbf{u}\|\,\|\nabla f\|\\
&\quad\leqslant \frac{(8)^{1/2}cC}{b}\left[\frac{a_2}{1-M}\right]^{1/2} D_2V^{1/2},
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
&a_2(\nabla\times\mathbf{u}\cdot\nabla w, f)\\
&\quad = -a_2(\nabla\times\mathbf{u}\cdot\nabla f, w) \leqslant (2)^{1/2}a_2\sup_{\Omega_1}|\mathbf{u}|\,\|\nabla\mathbf{u}\|\,\|\nabla f\|\\
&\quad\leqslant (2)^{1/2}a_2C\|\Delta\mathbf{u}\|\,\|\nabla\mathbf{u}\|\,\|\nabla f\| \leqslant \frac{4C}{b}\left[\frac{a_2}{1-M}\right]^{1/2} D_2V^{1/2}.
\end{aligned} \tag{4.18}$$

From (4.14)–(4.18) we get (4.12) with A given by (4.13).

Collecting (4.1), (4.5), (4.10), (4.11), (4.12), and taking into account Observation (1), we obtain the a priori estimate

$$\dot{V} \leqslant -D_2(1 - AV^{1/2}). \tag{4.19}$$

5. Nonlinear Stability Theorem

By virtue of Estimate (4.19) we are now in a position to prove the following theorem of nonlinear stability.

Theorem 1. *Let*

$$0 < M < 1, \tag{5.1}$$

$$V(0) < A^{-2} \tag{5.2}$$

with A given by (4.13). *Then*

$$V(t) \leqslant V(0) \exp\left\{\frac{-\pi^2(1-M)}{P^*}\left[1 - AV(0)^{1/2}\right]t\right\}, \tag{5.3}$$

where

$$P^* = \begin{cases} P_r & \text{for} \quad P_r \geqslant 1 \\ 1 & \text{for} \quad P_r < 1. \end{cases} \tag{5.4}$$

Proof. Conditions (5.1), (5.2), and Inequality (4.19) assure

$$\dot{V}(0) < 0. \tag{5.5}$$

Therefore, from (4.19), by a recursive argument, we obtain

$$\dot{V} \leqslant -D_2[1 - AV(0)^{1/2}] \qquad \forall t \geqslant 0. \tag{5.6}$$

Since, as is easily seen,

$$\int_{\Omega_1} f \, d\Omega_1 = 0,$$

we have (see [5])

$$\|\nabla f\|^2 \geqslant \pi^2 \|f\|^2. \tag{5.7}$$

Moreover, from (4, 10), (4.2), (4.7), (5.7), (3.4), and (3.5) it follows that

$$\frac{1-M}{2} D_0 = \frac{1-M}{2} \left[\|\Delta w\|^2 + \frac{a_1}{P_r} \|\nabla \vartheta_z\|^2 + a_2 \|\nabla f\|^2 \right]$$

$$\geqslant \frac{1-M}{2} \pi^2 \left[\|\nabla w\|^2 + \frac{a_1}{P_r} \|\vartheta_z\|^2 + a_2 \|f\|^2 \right] \geqslant \frac{(1-M)\,\pi^2}{P^*} V_0,$$

$$\frac{bD_1}{2} \geqslant \frac{b\pi^2}{P^*} V_1 \geqslant \frac{(1-M)\,\pi^2}{P^*} bV_1. \tag{5.8}$$

Then (5.6) implies

$$\dot{V} \leqslant -\frac{(1-M)}{P^*} \pi^2 [1 - AV(0)^{1/2}]\, V;$$

hence (5.3) follows.

6. Variational Problem and Critical Rayleigh Number

Inequality (5.3) shows that the range of nonlinear stability is strictly connected to the values given by the maximum M defined in (4.4). The aim of this section is to evaluate the nonlinear critical Rayleigh number related to M. For this let us choose

$$c = \frac{TP_r\xi}{R}, \qquad \xi \in (0, 1) \tag{6.1}$$

$$a_2 = \frac{1}{1-\xi}. \tag{6.2}$$

Then the variational problem (4.4) becomes

$$\frac{M}{R} = \max_{\mathscr{H}} \frac{-(\Delta_1 \vartheta, w) + \tau(w_z, \vartheta_z) + \mu(\nabla \vartheta_z, \nabla f)}{\|\Delta w\|^2 + (a_1/P_r)\|\nabla \vartheta_z\|^2 + (1/(1-\xi))\|\nabla f\|^2}, \tag{6.3}$$

where

$$\mathscr{H} = \{w, \vartheta, f \in C^4(\hat{\Omega}) \text{ which are periodic in } x \text{ and } y \text{ and satisfy } (2.3)_{1\text{-}2} \text{ and } (3.10)\},$$

$$\tau = \frac{a_1}{P_r} - \frac{T^2 P_r \xi}{R^2}, \tag{6.4}$$

$$\mu = -\frac{\xi}{1-\xi} \frac{T}{R^2} (P_r - 1). \tag{6.5}$$

The existence of the maximum (6.3) can be proved by the constructing method of the calculus of variations (based on Fourier expansions); see [28].

It is found (see Appendix) that

$$\frac{M}{R}=\begin{cases}\dfrac{1}{2}\left\{\dfrac{P_r}{a_1}\left[\dfrac{\tau^2}{\pi^4}+\mu^2(1-\xi)\right]\right\}^{1/2} & \text{for} \quad \tau<-1/3, \tau>2/3 \\ \dfrac{1}{2}\left\{\dfrac{P_r}{a_1}\left[\dfrac{4}{27(1-\tau)\,\pi^4}+\mu^2(1-\xi)\right]\right\}^{1/2} & \text{for} \quad \tau\in[-1/3, 2/3].\end{cases} \tag{6.6}$$

If we set

$$\frac{M}{R}=\frac{1}{R_\varepsilon}, \tag{6.7}$$

the stability condition (5.1) is equivalent to

$$R^2<R_\varepsilon^2, \tag{6.8}$$

where

$$R_\varepsilon^2=\begin{cases}\dfrac{\pi^4}{\tau^2}\left\{2\tau+\left[4\tau^2+\dfrac{\tau^2}{\pi^4}\,T^2 g(\xi)\right]^{1/2}\right\} \\ \qquad \text{for} \quad \tau<-1/3, \tau>2/3 \\ \dfrac{27}{2}\,\pi^4(1-\tau)\left\{\tau+\left[\tau^2+\dfrac{T^2}{27(1-\tau)\,\pi^4}\,g(\xi)\right]^{1/2}\right\} \\ \qquad \text{for} \quad \tau\in[-1/3, 2/3],\end{cases} \tag{6.9}$$

and $g(\xi)$ is given by

$$g(\xi)=4P_r\xi-\frac{\xi^2}{1-\xi}(P_r-1)^2. \tag{6.10}$$

We observe that R_ε^2 depends on the parameters P_r and T^2 (related to the basic motion) and on the Lyapunov parameters ξ and τ. They will be chosen to maximize (6.9) and to assure that a_1, given by (6.4), is positive.

For fixed τ, it is easy to see that

$$R_\varepsilon^2(\xi_0, \tau)=\max_{\xi\in(0,1)} R_\varepsilon^2(\xi, \tau) \tag{6.11}$$

for

$$\xi_0 = \begin{cases} \dfrac{2}{P_r + 1} & \text{when} \quad P_r > 1 \\[2ex] \dfrac{2P_r}{P_r + 1} & \text{when} \quad 0 < P_r < 1. \end{cases} \tag{6.12}$$

Then

$$R_\varepsilon^2(\tau) = R_\varepsilon^2(\xi_0, \tau) = \begin{cases} \dfrac{\pi^4}{\tau^2}\left[2\tau + \left(4\tau^2 + \dfrac{4T^2 p_0^2 \tau^2}{\pi^4}\right)^{1/2}\right] \\ \qquad \text{for} \quad \tau < -1/3, \tau > 2/3 \\ \dfrac{27}{2}\pi^4(1-\tau)\left[\tau + \left(\tau^2 + \dfrac{4T^2 p_0^2}{27(1-\tau)\pi^4}\right)^{1/2}\right] \\ \qquad \text{for} \quad \tau \in [-\frac{1}{3}, \frac{2}{3}] \end{cases} \tag{6.13}$$

with

$$p_0^2 = \begin{cases} 1 & \text{when} \quad P_r > 1 \\ P_r^2 & \text{when} \quad 0 < P_r < 1. \end{cases} \tag{6.14}$$

Maximizing (6.11) with respect to τ, we have

$$R_\varepsilon^2 = \max R_\varepsilon^2(\tau) = \begin{cases} 27\pi^4(1-\tau^*)^2 & \text{for} \quad 0 \leqslant T^2 p_0^2 \leqslant 80\pi^4 \\ 6\pi^4\left[\left(1 + \dfrac{T^2 p_0^2}{\pi^4}\right)^{1/2} - 1\right] & \text{for} \quad T^2 p_0^2 > 80\pi^4, \end{cases} \tag{6.15}$$

with τ^* given by the solution of the cubic equation

$$(1-\tau)^2(1-2\tau) = \frac{T^2 p_0^2}{27\pi^4} \tag{6.16}$$

in the interval $[-1/3, 1/2]$.

Of course, R_ε^2 *is the critical Rayleigh number of the nonlinear stability.*

From the calculations of the maximum (6.3) it is also found (see Appendix) that there exists a "wave number" associated to τ by the relation

$$a^2 = (2 - 3\tau)\pi^2, \qquad \tau < 2/3. \tag{6.17}$$

Then, by (6.19), we deduce that the "critical wave number"

$$a_\varepsilon = (2 - 3\tau^*)^{1/2}\pi, \qquad \tau^* \in [-1/3, 1/2], \tag{6.18}$$

increases from $\pi/(2)^{1/2}$ to $(3)^{1/2}\pi$ for $0 \leqslant T^2 p_0^2 \leqslant 80\pi^4$ and then remains at this value for $T^2 p_0^2 \geqslant 80\pi^4$. This is in contrast with linear theory where a_c goes to infinity as $T^{1/3}$ when $P_r > 1$ (see [3, pp. 95]), but is in agreement with the experiments of [11] obtained for rigid boundary conditions and with the results of [5]. We finally observe that with the previous choice of the Lyapunov parameters ξ and τ, when we assume (6.8), the function V_0 is positive definite. In fact, from (6.4) we have

$$\frac{a_1}{P_r} = \tau + \frac{T^2 P_r \xi_0}{R^2} > \tau + \frac{T^2 P_r \xi_0}{R_\varepsilon^2} \geqslant \tau + \frac{T^2 p_0^2}{R_\varepsilon^2} \geqslant 1 - \tau \geqslant \frac{1}{2}.$$

7. Comparison between R_ε^2 and R_L^2 in the Case $P_r > 1$. A Linearization Principle

In order to compare our previous results of linear stability to the "classical" ones (see [3, pp. 94–95]), let us indicate by R_L^2 the critical Rayleigh number of linear stability in the case in which the onset of convection begins as stationary convection ($P \geqslant 1$).

The following result holds true:

Theorem 2. *If*

$$P_r > 1, \qquad T^2[0, 80\pi^4] \tag{7.1}$$

then

$$R_\varepsilon^2 = R_L^2 \qquad \textit{and} \qquad R_\varepsilon^2 \in \left[\frac{27}{4}\pi^4, 48\pi^4\right]. \tag{7.2}$$

The condition $R^2 \leqslant R_L^2$ *also assures nonlinear conditional stability* (5.3) *under the condition* (5.2) $V(0) < A^{-2}$.

Proof. In the case $P_r \geqslant 1$, Chandrasekhar [3] shows that overstability cannot occur and the principle of exchange of stability is valid (see [3, pp. 116–118]). Moreover, from Eqs. (130) and (131) of [3, p. 95], one has

$$R_L^2 = \frac{\pi^4}{x}\left[(1+x)^3 + \frac{T^2}{\pi^4}\right], \tag{7.3}$$

where x is given by the solution of the cubic equation

$$2x^3 + 3x^2 = 1 + \frac{T^2}{\pi^4}, \qquad x \geqslant 1/2, \tag{7.4}$$

and is related to the linear wave number a_c by the formula

$$a_c^2 = \pi^2 x. \tag{7.5}$$

(Here the Taylor number is denoted by T^2 while in [3] it is denoted by T. This is due to the different nondimensional forms of the equations).

(7.3) and (7.4) imply

$$R_L^2 = 3\pi^4(1+x)^2. \tag{7.6}$$

Now, putting

$$a_\varepsilon^2 = \pi^2 y, \tag{7.7}$$

from (6.18), (6.16), and $(6.15)_1$ we obtain

$$R_\varepsilon^2 = 3\pi^4(1+y)^2, \qquad 1/2 \leqslant y \leqslant 3, \tag{7.8}$$

with y given by the solution of the equation

$$2y^3 + 3y^2 = 1 + \frac{T^2}{\pi^4}, \qquad T^2 \in [0, 80\pi^4]. \tag{7.9}$$

Hence, from (7.4), (7.6), and (7.8)–(7.9), we obtain (7.2). Then from the condition $R^2 < R_L^2$, (5.2), and Theorem 1, we deduce the non-linear asymptotic stability (5.3).

The values of the critical Rayleigh numbers R_ε^2 are shown in Table I.

TABLE I

Critical Rayleigh Numbers and Wave Numbers against Taylor Numbers T^2 in the Case $P_r > 1$

T^2	a_ε	R_ε^2	a_c	R_L^2	R_{G-S}^2
0	2.2214	657.511	2.2214	657.511	657.511
10	2.27	677.1	2.27	677.1	658.16
50	2.43	748.3	2.43	748.3	657.832
10^2	2.59	826.3	2.59	826.3	725.65
5×10^2	3.28	1,275	3.28	1,275	1,182.241
10^3	3.71	1,676	3.71	1,676	1,623.21
2×10^3	4.22	2,299	4.22	2,299	2,293.489
2630.05	4.44	2,630.05	4.44	2,630.05	2,630.05
5×10^3	5.01	3,670	5.01	3,670	3,626.32
7792	5.44	4,675	5.44	4,675	4,526
10^4	5.44	5,366	5.698	5,377	5,128
10^5	5.44	18,151	8.626	21,310	16,217
10^6	5.44	58,636	12.86	92,220	51,284
10^7	5.44	186,679	19.02	4.147×10^5	162,174
10^8	5.44	591,592	28.02	1.897×10^6	512,839

They are compared with those of Chandrasekhar [3], R_L^2, and Galdi and Straughan [5], R_{G-S}^2. For the sake of completeness the values of the critical wave numbers of the linear stability a_c and of the Lyapunov stability a_ε are also given. We observe that for $T^2 = 27\pi^4 = 2630.05$ it is found that $R_\varepsilon^2 = R_L^2 = R_{G-S}^2 = 27\pi^4$; this is not a special physical case but only a mathematical result depending on the choice of the Lyapunov functions.

8. On the Behaviour of the Attracting Radius for the Initial Data

In Section 5 we proved that Conditions (5.1) and (5.2) imply nonlinear stability (5.3). We can write (5.3) in the form

$$\|\nabla w(0)\|^2 + a_1 \|\vartheta_z(0)\|^2 + a_2 \|f(0)\|^2 + b[\|\nabla u(0)\|^2 + P_r \|\nabla \vartheta(0)\|^2] < \frac{b}{4C^2} A_1 \tag{8.1}$$

where all fields are evaluated for $t = 0$,

$$A_1 = \left\{ (1 + P_r)^{1/2}/2 + \frac{1}{(b - bM)^{1/2}} [2^{1/2} + (P_r a_1)^{1/2} + (a_2)^{1/2}(c + 2^{1/2})] \right\}^{-2}, \tag{8.2}$$

and C is a constant depending on the periodicity cell: $C = C(h)$, where $h = \min(1, a_1, a_2)$, with $2a_1$ and $2a_2$ the periods in x and y respectively (see [5, (A.15)]). Then $r_\varepsilon = bA_1/4C^2$ defines a set of initial values which are attracted to the basic motion m_0. According to [26, Chap. 1, p. 9] we call r_ε an *attracting radius* for the conditionally stable disturbances of m_0. Taking into account Definitions (4.9), (6.1)(6.2), (6.4), (6.7) we see that for any fixed R^2 such that $R^2 < R_\varepsilon^2$, we have that $A_1 = A_1(T^2)$ is an increasing function of T^2. Thus, from (8.1)–(8.2) it follows that the initial data need only be bounded by a known function of T^2. Moreover, since C is a computable constant, we have an exact known bound for the size of the attracting radius for the initial data. Now, in Table II, we compute—in the case of water ($P_r = 6.8$), for fixed Rayleigh numbers R^2 and for different Taylor numbers T^2—the size of the attracting radius in our case, r_ε, and in the Galdi and Straughan case, r_{G-S}. We observe that, in doing these computations, we may assume that $C(h) = C(1) = 10$, because if $a_i < 1$ $(i = 1, 2)$ one may consider a suitable "multi-cell." With this choice $r_\varepsilon = bA_1/400$, with b and A_1 given by (4.9) and (8.2); $r_{G-S} = 2A^{-2}$, with A given by (5.36) of [5].

TABLE II

Values of the Attracting Radius in the Case of Water for Different R^2 and T^2

	T^2	r_ε	r_{G-S}
$R^2 = 625$			
	0	2.2×10^{-13}	
	50	3.9×10^{-11}	1.7×10^{-13}
	10^2	4.5×10^{-11}	6.3×10^{-13}
	10^3	6×10^{-9}	5.3×10^{-10}
	10^4	5×10^{-7}	3.3×10^{-8}
	10^5	7.3×10^{-6}	6.5×10^{-7}
$R^2 = 1000$			
	10^9	2.5×10^{-10}	1.3×10^{-11}
	10^4	5.1×10^{-8}	3.3×10^{-9}
	10^5	1.6×10^{-6}	8×10^{-8}
$R^2 = 2000$			
	10^4	2×10^{-9}	6.4×10^{-11}
	10^5	1.3×10^{-7}	3.5×10^{-9}

9. Some Final Comments

We make here some comments on the choice of the Lyapunov function and on the stability results obtained. As far as we know, in the theory of fluid motion stability, Lyapunov functions—more general than the energy norm—were used first by Pritchard [15] and successively by Joseph [19]. But the first use of a *balance* field in a Lyapunov function was by Joseph [19] in the case of heated and salty-below-conduction-diffusion problem. The balance field f has also been considered in the Lyapunov function used in [5]. But neither [19] nor [5] uses the essential variables. However, we observe that the function (3.4)–(3.5) is simpler than the one which is used in [5] and moreover it seems more physically motivated [see 17–18].

Concerning the values of the attracting radius for the initial data, we observe that, although better than those of [5], they are very low. Of course we cannot exclude the possibility that, by using another Lyapunov function, one can reach *better* values. However, we remark that, at least in the range $0 \leqslant T^2 \leqslant 80\pi^4$, $27\pi^4/4 \leqslant R^2 \leqslant 48\pi^4$, the validity of the linearization principle is proved under explicit values of attracting radius for the initial data.

APPENDIX

Here we prove (6.6).

The Euler–Lagrange equations associated with (6.3) are

$$\begin{aligned}&\Delta f=\mu\rho(1-\xi)\,\Delta\vartheta_z/2\\ &\Delta\Delta w=-\rho(\Delta_1\vartheta+\tau\vartheta_{zz})/2\\ &\Delta_1 w+\tau w_{zz}-\mu\,\Delta f_z+\frac{2a_1}{\rho P_r}\,\Delta\vartheta_{zz}=0,\end{aligned} \tag{A.1}$$

where $1/\rho$ is the generic eigenvalue related to the maximum problem (6.3). To solve (A.1) we differentiate $(A.1)_1$ with respect to z and substitute the value of Δf_z so obtained in $(A.1)_3$. Moreover we take the double Δ of the equation so deduced and use $(A.1)_2$. Thus we find

$$\Delta_1\Delta_1\vartheta+2\tau\Delta_1\vartheta_{zz}+\tau^2\vartheta_{zzzz}+\mu^2(1-\xi)\,\Delta\Delta\Delta\vartheta_{zz}-\frac{4a_1}{\rho^2P_r}\,\Delta\Delta\Delta\vartheta_{zz}=0. \tag{A.2}$$

As this equation is linear, we seek solutions of the form

$$\vartheta=\theta(z)\exp i(m\alpha_1x+p\alpha_2y),\qquad \alpha_1>0,\alpha_2>0,m,\,p\in\mathbb{N}. \tag{A.3}$$

The boundary conditions that $\theta(z)$ must satisfy may be obtained as in [3]:

$$\theta(z)=\theta^{(2k)}(z)=0\qquad \text{on } z=0,z=1,k=1,2,\ldots. \tag{A.4}$$

Hence it is sufficient to consider $\theta(z)$ of the form

$$\theta(z)=A_0\sin n\pi z,\qquad n=1,2,\ldots,A_0\neq 0 \tag{A.5}$$

With (A.3) and (A.5), (A.2) may be solved for $1/\rho^2$ to obtain

$$1/\rho^2=\frac{P_r}{4a_1\pi^2}\left[\frac{(m^2\alpha_1^2+p^2\alpha_2^2+\tau n^2\pi^2)^2}{(m^2\alpha_1^2+p^2\alpha_2^2+n^2\pi^2)^3}+\pi^2\mu^2(1-\xi)\right]. \tag{A.6}$$

Therefore, we have

$$(M/R)^2=\max 1/\rho^2 \tag{A.7}$$

as m, p, n, α_1, and α_2 assume the values $m,\ p=0,1,2,\ldots,n=1,2,\ldots,\alpha_1,\alpha_2\in\mathbb{R}$.

In order to obtain (A.7) it is sufficient to study the function

$$g(X,Y,Z)=\frac{(\alpha_1^2X+\alpha_2^2Y+\tau\pi^2Z)^2}{Z(\alpha_1^2X+\alpha_2^2Y+\pi^2Z)^3}$$

in $\Omega_0 = [0, +\infty) \times [0, +\infty) \times [1, +\infty)$. It is found that

$$\max_{\alpha_1, \alpha_2} \max_{\Omega_0} g(X, Y, Z) = \begin{cases} \dfrac{\tau^2}{\pi^2} & \text{for} \quad \tau < -1/3, \tau > 2/3 \\ \dfrac{4}{27(1-\tau)\pi^2} & \text{for} \quad \tau \in [-1/3, 2/3]. \end{cases}$$

In the case $\tau \in [-1/3, 2/3]$, for $m = p = 1$, we have the following relation between $a^2 = \alpha_1^2 + \alpha_2^2$ and τ:

$$a^2 = (2 - 3\tau)\,\pi^2. \tag{A.8}$$

(A.8) can be obtained by putting equal to zero the system of the first derivatives of $g(X, Y, Z)$ with respect to X, Y, Z.

Acknowledgments

This research has been performed under the auspices of the G.N.F.M. of the C.N.R. and has been partially supported by the Italian Ministry for Education (M.P.I.) under 40% and 60% contracts and by C.N.R.-S.P.A.I.M. The authors thank G. P. Galdi and B. Straughan for sending a copy of the proofs of their paper in advance of publication.

References

1. S. K. Chan, Investigation on turbulent convection under a rotational constraint, *J. Fluid Mech.* **64** (1974), 477–506.
2. S. Chandrasekhar, The instability of a layer of fluid heated below and subject to Coriolis forces, *Proc. Roy. Soc. London Ser. A* **217** (1953), 306–327.
3. S. Chadrasekhar, "Hydrodynamic and Hydromagnetic Stability," Dover, New York, 1981.
4. S. Chandrasekhar and D. D. Elbert, The stability of a layer of fluid heated below and subject to Coriolis forces, II, *Proc. Roy. Soc. London Ser. A* **231** (1955), 198–210.
5. G. P. Galdi and B. Straughan, A nonlinear analysis of the stabilizing effect of rotation in the Bénard problem, *Proc. Roy. Soc. London Ser. A* **402** (1985), 257–283.
6. C. Hunter and N. Riahi, Nonlinear convection in a rotating fluid, *J. Fluid Mech.* **72** (1975), 433–454.
7. P. Kloeden and R. Wells, An explicit example of Hopf bifurcation of fluidmechanics, *Proc. Roy. Soc. London Ser. A* **390** (1983), 293–320.
8. J. C. Morgan, Nonlinear Bénard convection with rotation, *J. Fluid Mech.* **57** (1973), 433–458.
9. Y. Nakagawa and P. Frenzen, A theoretical and experimental study of cellular convection in rotating fluids, *Tellus* **7** (1955), 1–21.
10. P. P. Niiler and F. E. Bisshopp, On the influence of Coriolis force on onset of thermal convection, *J. Fluid Mech.* **22** (1965), 753–761.
11. H. T. Rossby, A study of Bénard convection with and without rotation, *J. Fluid Mech.* **36** (1969), 309–335.

12. G. Veronis, Cellular convection with finite amplitude in a rotating fluid, *J. Fluid Mech.* **5** (1959), 401–435.
13. G. Veronis, Motions at subscritical values of the Rayleigh number in a rotating fluid. *J. Fluid Mech.* **24** (1966), 545–554.
14. G. Veronis, Large amplitude Bénard convection in a rotating fluid, *J. Fluid Mech.* **31** (1968), 113–139.
15. A. J. Pritchard, A study of the classical problems of hydrodynamic stability by the Lyapunov method, *J. Inst. Math. Appl.* **4** (1968), 78–93.
16. G. P. Galdi, Non-linear stability of the magnetic Bénard problem via a generalized energy method, *Arch. Rational Mech. Anal.* **87** (1985), 167–186.
17. S. Rionero, On the choice of the Lyapunov function in the fluid motion stability, lecture given at the meeting "Energy stability and convection," Capri, May 1986.
18. S. Rionero and G. Mulone, A nonlinear stability analysis of the magnetic Bénard problem via the Lyaponov direct method, *Arch. Rational Mech. Anal* **103** (1988), 347–368.
19. D. D. Joseph, Global stability of the conduction–diffusion solution, *Arch. Rational Mech. Anal.* **36** (1970), 285–292.
20. G. Prodi, Teoremi di tipo locale per il sistema di Navier-Stokes *e* stabilità delle soluzioni stazionarie, *Rend. Sem. Mat. Univ. Padova* **32** (1962), 374–397.
21. O. A. Ladyzhenskaya and V. A. Solonnikov, The linearization principle and invariant manifolds for problems of magnetohydrodynamics, *J. Sov. Math.* **8** (1977), 384–422.
22. G. Mulone and F. Salemi, On the hydrodynamic motion in a domain with mixed boundary conditions: Existence, uniqueness, stability and linearization principle, *Ann. Mat. Pura App.* **139** (1985), 147–174.
23. S. Rionero and G. Mulone, On the nonlinear stability of a thermo-diffusive fluid mixture in a mixed problem, *J. Math. Anal. Appl.* **124** (1987), 165–188.
24. G. H. Hardy, J. E. Littlewood, and G. Polya, "Inequalities," Cambridge Univ. Press, 1959.
25. R. A. Adams, "Sobolev spaces," Academic Press, New York, 1975.
26. O. A. Ladyzhenskaya, "The mathematical theory of viscous incompressible flow," Gordon & Breach, New York, 1975.
27. D. D. Joseph, "Stability of Fluid Motions I," Springer-Verlag, Berlin/New York, 1976.
28. S. Rionero and G. Mulone, On a maximum problem governing the nonlinear stability of rotating Bénard problem, *Ricerche di Mat.* (in press).